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Project
TMA4265 Stochastic processes

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Abstract

In this project we find, simulate and discuss various probabilities and expected values for the game yahtzee. `MATLAB` has been used for all simulations and most of the calculations of the analytic reults. We will find that 10 000 simulations of 100-roll Markov chains yield good results compared to the analytic values.

Task 1

State 0 can be omitted because you will never enter state 0 again, after you have rolled the first roll. And the probabilities for going from state 0 to any of the other states is equal to the probabilities for going from state 1 to any of the other states. We therefore remove the first row and column in the transition matrix given in Problem 1 of the TMA4265 exam of 2008 Autumn, and obtain:

$$P = \begin{bmatrix} \frac{5}{54} & \frac{25}{36} & \frac{125}{648} & \frac{25}{1296} & \frac{1}{1296} \\ 0 & \frac{5}{9} & \frac{10}{27} & \frac{5}{72} & \frac{1}{216} \\ 0 & 0 & \frac{25}{36} & \frac{5}{18} & \frac{1}{36} \\ 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We are now left with states 1, 2, 3, 4 and 5. Here state 5 is absorbing.

To calculate the transition probabilities P_{12}, P_{13}, P_{22} and P_{23}, we consider favourable outcomes divided by all possible outcomes.

$$\begin{aligned} P_{12} &= P(1 \text{ pair}, 3 \text{ different}) + P(2 \text{ pairs}, 1 \text{ different}) \\ &= \frac{\binom{6}{1}\binom{5}{2}\binom{5}{1}\binom{4}{1}\binom{3}{1}}{6^5} + \frac{\binom{6}{2}\binom{5}{2}\binom{3}{2}\binom{4}{1}}{6^5} = \frac{25}{36} \end{aligned}$$

$$\begin{aligned} P_{13} &= P(3 \text{ equal}, 2 \text{ different}) + P(3 \text{ equal}, 1 \text{ pair}) \\ &= \frac{\binom{6}{1}\binom{5}{3}\binom{5}{1}\binom{4}{1}}{6^5} + \frac{2 \cdot \binom{6}{2}\binom{5}{3}\binom{2}{2}}{6^5} = \frac{125}{648} \end{aligned}$$

$$\begin{aligned} P_{22} &= P(3 \text{ different}) + P(1 \text{ pair}, 1 \text{ different}) \\ &= \frac{\binom{5}{1}\binom{4}{1}\binom{3}{1}}{6^3} + \frac{\binom{5}{1}\binom{3}{2}\binom{4}{1}}{6^3} = \frac{5}{9} \end{aligned}$$

$$\begin{aligned} P_{23} &= P(1 \text{ equal to those already saved}, 2 \text{ different}) \\ &\quad + P(1 \text{ equal to those already saved}, 1 \text{ pair}) \\ &\quad + P(3 \text{ equal, but not equal to those already saved}) \\ &= \frac{\binom{3}{1}\binom{5}{1}\binom{4}{1}}{6^3} + \frac{\binom{3}{1}\binom{5}{1}\binom{2}{2}}{6^3} + \frac{\binom{5}{1}}{6^3} = \frac{10}{27} \end{aligned}$$

Task 2

The `MATLAB` code can be found in the enclosed files `markovDie.m` and `simulation10k.m`. Note that these need to be in the same directory to run properly.

Task 3

To obtain the expected number of rolls needed to get five-of-a-kind for the first time we can use equation (1) derived in [Ross(2007), p. 228].

$$S = (I - P_T)^{-1} \tag{1}$$

Here

$$P_T = \begin{bmatrix} \frac{5}{54} & \frac{25}{36} & \frac{125}{648} & \frac{25}{1296} \\ 0 & \frac{5}{9} & \frac{10}{27} & \frac{5}{72} \\ 0 & 0 & \frac{25}{36} & \frac{5}{18} \\ 0 & 0 & 0 & \frac{5}{6} \end{bmatrix},$$

namely the 4×4 matrix of transient states in the transition matrix P.
From equation (1) we obtain

$$S = \begin{bmatrix} 1.1020 & 1.7219 & 2.7829 & 5.4832 \\ 0 & 2.2500 & 2.7273 & 5.4830 \\ 0 & 0 & 3.2727 & 5.4545 \\ 0 & 0 & 0 & 6.0000 \end{bmatrix},$$

which contains the expected time spent in state j given that we started in state i. Since we are interested in the time spent in all consecutive states given that we started in state 1, we take the sum of the first row; yielding that the expected number of throws to obtain five-of-a-kind is 11.09. From the simulation we get 11.08 with a standard deviation of 0.0641, so this is reasonable. The standard deviation for each simulation is 6.41.

The expected number of rolls to obtain five-of-a-kind if we alredy have three-of-a-kind, is obtained in the same way, but by taking the sum of the third row instead; yielding 8.73.
Simululation returns 8.72 with a standard deviation of 0.0715, which confirms the calculation. The standard deviation for each simulation is 6.61.

The expected number of rolls needed to obtain three-of-a-kind or more is obtained by the sum

$$S_{11} + S_{12} = 1.1020 + 1.7219 = 2.82.$$

The same result can be obtained by making the states 3, 4 and 5 absorbing, calculate the matrix S and sum the top row.
From the simulation we get 2.80 with a standard deviation of 0.0174. The standard deviation for each simulation is 1.74.

Task 4

To find the probability that state three occurs we can use equation (2) derived in [Ross(2007), p. 232], with $i = 1$ and $j = 3$

$$f_{ij} = \frac{s_{ij} - \delta_{i,j}}{s_{ij}}. \tag{2}$$

Hence the probability that state 3 is ever entered when we start in state 1 is

$$f_{13} = \frac{s_{13} - \delta_{1,3}}{s_{13}} = \frac{2.7829 - 0}{3.2727} = 0.85. \tag{3}$$

Confirmed by the probability 0.85 from the simulation.
Considering that the probability of going directly from state 2 to state 4 is $P_{24} = 0.07$ and that the other probabilities of not entering state 3 is even lower, it is reasonable that the probability of going via state 3 is this high.

Task 5

Assuming that all dice are re-rolled when five-of-a-kind is obtained; the new transition matrix is:

$$P_{new} = \begin{bmatrix} \frac{5}{54} & \frac{25}{36} & \frac{125}{648} & \frac{25}{1296} & \frac{1}{1296} \\ 0 & \frac{5}{9} & \frac{10}{27} & \frac{5}{72} & \frac{1}{216} \\ 0 & 0 & \frac{25}{36} & \frac{5}{18} & \frac{1}{36} \\ 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ \frac{5}{54} & \frac{25}{36} & \frac{125}{648} & \frac{25}{1296} & \frac{1}{1296} \end{bmatrix}$$

Because we now have an irreducible ergodic Markov chain, the limiting probabilities exists. These can be found by solving the linear system:

$$\pi_j = \sum_{i=1}^{5} \pi_i P_{ij}, \qquad j = 1,\ 2, \cdots,\ 5, \qquad \sum_{j=1}^{5} \pi_j = 1 \tag{4}$$

This system can be written as

$$\begin{bmatrix} -\frac{49}{54} & 0 & 0 & 0 & \frac{5}{54} \\ \frac{25}{36} & -\frac{4}{9} & 0 & 0 & \frac{25}{36} \\ \frac{125}{648} & \frac{10}{27} & -\frac{9}{36} & 0 & \frac{125}{648} \\ \frac{25}{1296} & \frac{5}{72} & \frac{5}{18} & -\frac{1}{6} & \frac{25}{1296} \\ \frac{1}{1296} & \frac{1}{216} & \frac{1}{36} & \frac{1}{6} & -\frac{1295}{1296} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{bmatrix} = \begin{bmatrix} 0.0092 \\ 0.1553 \\ 0.2509 \\ 0.4944 \\ 0.0902 \end{bmatrix}$$

Comparing with simulation which gives,

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{bmatrix} = \begin{bmatrix} 0.0092 \\ 0.1564 \\ 0.2534 \\ 0.4904 \\ 0.0907 \end{bmatrix},$$

the answer is confirmed.

Task 6

The probability for obtaining five-of-a-kind in three or fewer rolls, is found by computing the third power of the transition matrix and taking element $P^3_{15} = 0.0460$. (Computed in `MATLAB`.) The simulation gives the probability 0.0452.

Generally P^n_{ij} is the probability for going from state i to state j in n steps. Here state 5 is absorbing, so this also includes fewer than n steps if $j = 5$.

The probability for obtaining five-of-a-kind in three or fewer rolls if you obtain two-of-a-kind in the first roll, is computed by taking the second power of the transition matrix and looking at element $P^2_{25} = 0.0291$. (Computed in `MATLAB`.) The simulation yields the probability 0.0280.

From this we can conclude that we should not be happy if we obtain two-of-a-kind in the first roll since $P^2_{25} < P^3_{15}$.

Task 7

Plots for the cummulative distributions can be found in Figure 1. As expected we note that more simulations yield a smoother curve.

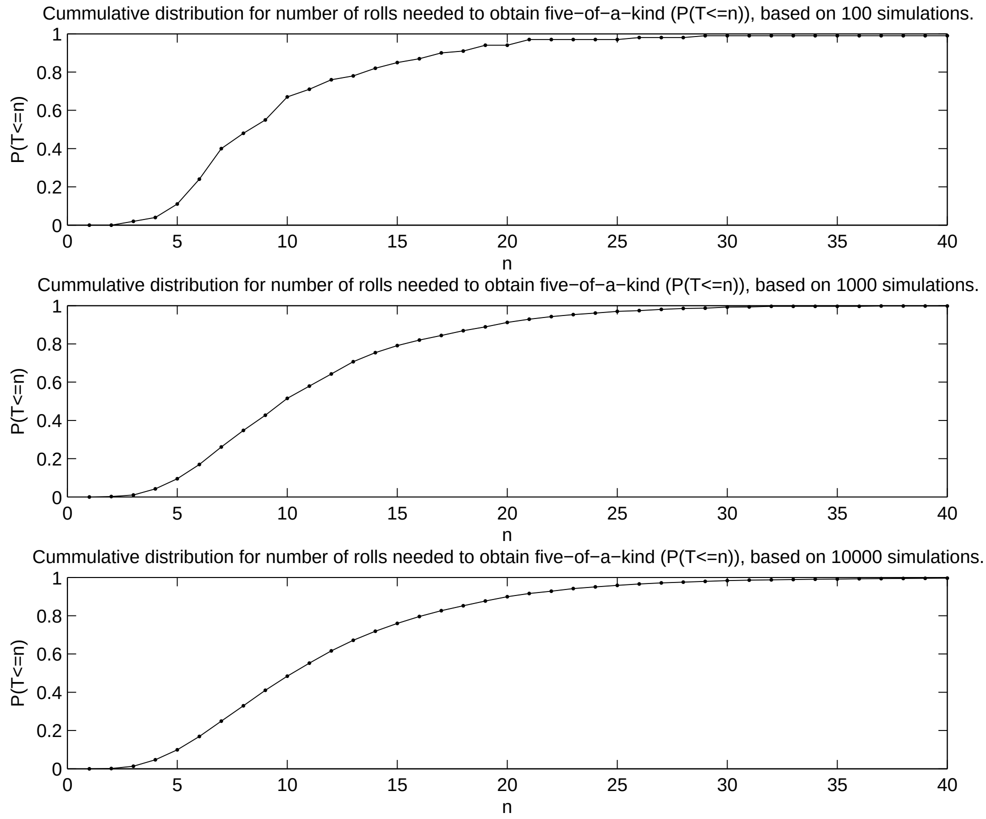

Figure 1: Cummualitve distributions for number of rolls needed to obtain five-of-a-kind based on 100, 1000 and 10000 simulations. The plots have been cut off after 40 rolls, since the cummulative distribution has no significant changes after that.

References

[1] Sheldon M. Ross. *Introduction to Probability Models.* Academic Press, University of California, Berkeley, California, 9th, edition, 2007.